

Chapter 8. Binomial Theorem

Question-1

Expand each of the following expression $(1 - 2x)^5$

Solution:

By using Binomial Theorem, we have

$$\begin{aligned}(1 - 2x)^5 &= [1 + (-2x)]^5 \\&= {}^5C_0 + {}^5C_1(-2x) + {}^5C_2(-2x)^2 + {}^5C_3(-2x)^3 + {}^5C_4(-2x)^4 + {}^5C_5(-2x)^5 \\&= 1 + 5(-2x) + 10(-2x)^2 + 10(-2x)^3 + 5(-2x)^4 + (-2x)^5 \\&= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5\end{aligned}$$

Question-2

Expand each of the following expression $\left(\frac{2}{x} - \frac{x}{2}\right)^5$

Solution:

By using Binomial Theorem, we have

$$\begin{aligned}\left(\frac{2}{x} - \frac{x}{2}\right)^5 &= \left[\frac{2}{x} + \left(-\frac{x}{2}\right)\right]^5 \\&= {}^5C_0\left(\frac{2}{x}\right)^5\left(-\frac{x}{2}\right)^0 + {}^5C_1\left(\frac{2}{x}\right)^4\left(-\frac{x}{2}\right)^1 + {}^5C_2\left(\frac{2}{x}\right)^3\left(-\frac{x}{2}\right)^2 + {}^5C_3\left(\frac{2}{x}\right)^2\left(-\frac{x}{2}\right)^3 + {}^5C_4\left(\frac{2}{x}\right)^1\left(-\frac{x}{2}\right)^4 + {}^5C_5\left(\frac{2}{x}\right)^0\left(-\frac{x}{2}\right)^5 \\&= \frac{32}{x^5} + 5\left(\frac{16}{x^4}\right)\left(-\frac{x}{2}\right) + 10\left(\frac{8}{x^3}\right)\left(\frac{x^2}{4}\right) + 10\left(\frac{4}{x^2}\right)\left(-\frac{x^3}{8}\right) + 5\left(\frac{2}{x}\right)\left(\frac{x^4}{16}\right) + \left(-\frac{x^5}{32}\right) \\&= \frac{32}{x^5} - \frac{40}{x^3} + \frac{20}{x} - 5x + \frac{5x^2}{8} - \frac{x^5}{32}\end{aligned}$$

Question-3

Expand each of the following expression $(2x - 3)^6$

Solution:

By using Binomial Theorem, we have

$$\begin{aligned}(2x - 3)^6 &= [2x + (-3)]^6 \\&= {}^6C_0(2x)^6(-3)^0 + {}^6C_1(2x)^5(-3)^1 + {}^6C_2(2x)^4(-3)^2 + {}^6C_3(2x)^3(-3)^3 + {}^6C_4(2x)^2(-3)^4 + {}^6C_5(2x)^1(-3)^5 + {}^6C_6(2x)^0(-3)^6 \\&= 1(2x)^6 + 6(2x)^5(-3) + 15(2x)^4(-3)^2 + 20(2x)^3(-3)^3 + 15(2x)^2(-3)^4 + 6(2x)(-3)^5 + (-3)^6 \\&= 64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729\end{aligned}$$

Question-4

Expand each of the following expression $\left(\frac{x}{3} + \frac{1}{x}\right)^5$

Solution:

By using Binomial Theorem, we have;

$$\begin{aligned}\left(\frac{x}{3} + \frac{1}{x}\right)^5 &= {}^5C_0 \left(\frac{x}{3}\right)^5 \left(\frac{1}{x}\right)^0 + {}^5C_1 \left(\frac{x}{3}\right)^4 \left(\frac{1}{x}\right)^1 + {}^5C_2 \left(\frac{x}{3}\right)^3 \left(\frac{1}{x}\right)^2 + {}^5C_3 \left(\frac{x}{3}\right)^2 \left(\frac{1}{x}\right)^3 + {}^5C_4 \\ &\left(\frac{x}{3}\right)^1 \left(\frac{1}{x}\right)^4 + {}^5C_5 \left(\frac{x}{3}\right)^0 \left(\frac{1}{x}\right)^5 \\ &= \left(\frac{x}{3}\right)^5 + 5\left(\frac{x}{3}\right)^4 \left(\frac{1}{x}\right) + 10\left(\frac{x}{3}\right)^3 \left(\frac{1}{x}\right)^2 + 10\left(\frac{x}{3}\right)^2 \left(\frac{1}{x}\right)^3 + 5\left(\frac{x}{3}\right) \left(\frac{1}{x}\right)^4 + \left(\frac{1}{x}\right)^5 \\ &= \frac{x}{243} + \frac{5x^3}{3} + \frac{10x}{3} + \frac{10}{3x} + \frac{5}{3x^3} + \frac{1}{x^5}\end{aligned}$$

Question-5

Expand each of the following expression $\left(x + \frac{1}{x}\right)^6$

Solution:

By using Binomial Theorem, we have:

$$\begin{aligned}\left(x + \frac{1}{x}\right)^6 &= {}^6C_0(x)^6\left(\frac{1}{x}\right)^0 + {}^6C_1(x)^5\left(\frac{1}{x}\right)^1 + {}^6C_2(x)^4\left(\frac{1}{x}\right)^2 + {}^6C_3(x)^3\left(\frac{1}{x}\right)^3 + {}^6C_4(x)^2\left(\frac{1}{x}\right)^4 + \\ &{}^6C_5(x)^1\left(\frac{1}{x}\right)^5 + {}^6C_6\left(\frac{1}{x}\right)^6 \\ &= x^6 + 6x^5\left(\frac{1}{x}\right) + 15x^4 \cdot \frac{1}{x^2} + 20x^3 \cdot \frac{1}{x^3} + 15x^2 \cdot \frac{1}{x^4} + 6x \cdot \frac{1}{x^5} + \frac{1}{x^6} \\ I &= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}\end{aligned}$$

Question-6

Using binomial theorem, evaluate each of the following $(96)^3$

Solution:

We express 96 as the sum or difference of two numbers whose powers are easier to calculate, and then use Binomial Theorem

Write $96 = 100 - 4$

Therefore

$$\begin{aligned}(96)^3 &= (100 - 4)^3 \\ &= {}^3C_0(100)^3 - {}^3C_1(100)^2(4) + {}^3C_2(100)^1(4)^2 - {}^3C_3(4)^3 \\ &= 1000000 - 3(10000)(4) + 3(100)(16) - (64) \\ &= 1000000 - 120000 + 4800 - 64 \\ &= 884736\end{aligned}$$

Question-7

Using binomial theorem, evaluate each of the following $(102)^5$

Solution:

$$\begin{aligned}(102)^5 &= (100 + 2)^5 \\&= {}^5C_0(100)^5 + {}^5C_1(100)^4(2) + {}^5C_2(100)^3(2)^2 + {}^5C_3(100)^2(2)^3 + {}^5C_4(100)(2)^4 + {}^5C_5(2)^5 \\&= 10000000000 + 5(100000000)(2) + 10(1000000)(4) + 10(10000)(8) + 5(100)(16) + 32 \\&= 10000000000 + 1000000000 + 40000000 + 800000 + 8000 + 32 \\&= 11040808032\end{aligned}$$

Question-8

Using binomial theorem, evaluate each of the following $(101)^4$

Solution:

$$\begin{aligned}(101)^4 &= (100 + 1)^4 \\&= {}^4C_0(100)^4 + {}^4C_1(100)^3(1) + {}^4C_2(100)^2(1)^2 + {}^4C_3(100)(1)^3 + {}^4C_4(1)^4 \\&= 100000000 + 4(1000000) + 6(10000) + 4(100) + 1 \\&= 100000000 + 4000000 + 60000 + 400 + 1 \\&= 104060401\end{aligned}$$

Question-9

Using binomial theorem, evaluate each of the following $(99)^5$

Solution:

$$\begin{aligned}(99)^5 &= (100 - 1)^5 \\&= {}^5C_0(100)^5 - {}^5C_1(100)^4(1) + {}^5C_2(100)^3(1)^2 - {}^5C_3(100)^2(1)^3 + {}^5C_4(100)(1)^4 - {}^5C_5(1)^5 \\&= (100)^5 - 5 \times (100)^4 + 10 \times (100)^3 - 10 \times (100)^2 + 5 \times 100 - 1 \\&= 10000000000 - 5000000000 + 100000000 - 100000 + 500 - 1 \\&= 10010000500 - 500100001 \\&= 9509900499\end{aligned}$$

Question-10

Using Binomial Theorem indicate which number is larger $(1.1)^{10000}$ or 1000.

Solution:

Splitting 1.1 and using Binomial Theorem to write the first few terms we have

$$\begin{aligned}(1.1)^{10000} &= (1+0.1)^{10000} \\&= {}^{10000}C_0 + {}^{10000}C_1(0.1) + {}^{10000}C_2(0.1)^2 + \text{other positive terms.} \\&= 1 + 10000 \cdot (0.1) + \text{other positive terms} \\&= 1 + 1000 + \text{other positive terms} \\&> 1000\end{aligned}$$

Hence, $(1.1)^{10000} > 1000$.

Question-11

Find $(a+b)^4 - (a-b)^4$. Hence, evaluate $(\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4$.

Solution:

$$\begin{aligned}(a+b)^4 - (a-b)^4 &= [{}^4C_0a^4 + {}^4C_1a^3b + {}^4C_2a^2b^2 + {}^4C_3ab^3 + {}^4C_4b^4] - [{}^4C_0a^4 - {}^4C_1a^3b + {}^4C_2a^2b^2 - {}^4C_3ab^3 + {}^4C_4b^4] \\&= 2 \times {}^4C_1a^3b + 2 \times {}^4C_3ab^3 \\&= 2[4a^3b + 4ab^3] \\&= 8ab[a^2 + b^2] \\ \text{Thus, } (\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4 &= 8 \cdot \sqrt{3} \cdot \sqrt{2} [3 + 2] = 8\sqrt{6} (5) = 40\sqrt{6}\end{aligned}$$

Question-12

Find $(x+1)^6 + (x-1)^6$. Hence or otherwise evaluate $(\sqrt{2} + 1)^6 - (\sqrt{2} - 1)^6$

Solution:

$$\begin{aligned}(x+1)^6 + (x-1)^6 &= [{}^6C_0x^6 + {}^6C_1x^5 + {}^6C_2x^4 + {}^6C_3x^3 + {}^6C_4x^2 + {}^6C_5x + {}^6C_6] + [{}^6C_0x^6 - {}^6C_1x^5 + {}^6C_2x^4 - {}^6C_3x^3 + {}^6C_4x^2 - {}^6C_5x + {}^6C_6] \\&= 2[{}^6C_0x^6 + {}^6C_2x^4 + {}^6C_4x^2 + {}^6C_6] \\&= 2[x^6 + 15x^4 + 15x^2 + 1] \\ \text{Thus, } (\sqrt{2} + 1)^6 + (\sqrt{2} - 1)^6 &= 2[(\sqrt{2})^6 + 15(\sqrt{2})^4 + 15(\sqrt{2})^2 + 1] \\&= 2[8 + 15(4) + 15(2) + 1] \\&= 2[8 + 60 + 30 + 1] = 198.\end{aligned}$$

Question-13

Show that $9^{n+1} - 8n - 9$ is divisible by 64, whenever n is a positive integer.

Solution:

$$n = 1 \Rightarrow 9^{n+1} - 8n - 9 = 9^2 - 8 - 9$$

$$= 81 - 17 = 64 = 1(64)$$

$$n = 2 \Rightarrow 9^{n+1} - 8n - 9 = 9^3 - 8(2) - 9$$

$$= 729 - 16 - 9 = 704 = 11(64)$$

$$\text{From } n = 3, 4, 5, \dots, 9^{n+1} - 8n - 9 = 9(1 + 8)n - 8n - 9$$

$$= 9 [{}^nC_0 + {}^nC_1 \cdot 8 + {}^nC_2 \cdot 8^2 + \dots + {}^nC_n 8^n] - 8n - 9$$

$$= 9[1 + 8n + {}^nC_2 \cdot 8^2 + \dots + {}^nC_n 8^n] - 8n - 9$$

$$= 9 + 72n + 9 \cdot {}^nC_2 \cdot 8^2 + \dots + 9 \cdot {}^nC_n 8^n - 8n - 9$$

$$= 8^2 [n + 9 ({}^nC_2 + {}^nC_3 \cdot 8 + \dots + {}^nC_n 8^{n-2})]$$

which is divisible by 64.

Question-14

Prove that $\sum_{r=0}^n 3^r {}^nC_r = 4^n$.

Solution:

$$\text{L.H.S} = 3^0 C(n,0) + 3^1 C(n,1) + 3^2 C(n,2) + \dots + 3^r C(n,r) + \dots + 3^n C(n,n)$$

$$= C(n,0) + C(n,1) 3^1 + C(n,2) 3^2 + C(n,3) 3^3 + \dots + C(n,n) 3^n$$

This is in the form of $(1+3)^n$

$$= (1+3)^n = 4^n = \text{R.H.S}$$

Question-15

Prove that x^5 in $(x + 3)^8$

Solution:

Suppose x^5 occurs in the $(r + 1)$ th term of the expansion $(x + 3)^8$

$$\text{Now } T_{r+1} = {}^nC_r a^{n-r} b^r = {}^8C_r x^{8-r} 3^r$$

Comparing the indices of x in x^5 and in T_{r+1} , we get $r = 3$

Thus, the coefficient of x^5 is

$${}^8C_3(3)^3 = 1512$$

Question-16

Prove that a^5b^7 in $(a-2b)^{12}$

Solution:

Let a^5b^7 occurs in the $(r+1)$ th term, in the expansion of $(a-2b)^{12}$ given by ${}^{12}C_r \cdot a^{12-r}(-2b)^r$. Then $12-r=5$. This gives $r=7$.

Thus the coefficient of a^5b^7 is

$${}^{12}C_5 (-2)^7 = \frac{12!}{5!7!} \times (-128) = (792) (-128) = -101376$$

Question-17

Prove that $(x^2 - y)^6$

Solution:

We have T_{r+1} in $(a+b)^n = {}^nC_r a^{n-r} \cdot b^r, 0 \leq r \leq n$

$$\begin{aligned} T_{r+1} \text{ in } (x^2 - y)^6 &= {}^6C_r (x^2)^{6-r} (-y)^r \\ &= {}^6C_r x^{12-2r} (-y)^r \end{aligned}$$

Question-18

Prove that $(x^2 - yx)^{12}, x \neq 0$

Solution:

$$\begin{aligned} T_{r+1} \text{ in } (x^2 - yx)^{12} &= {}^{12}C_r (x^2)^{12-r} (-yx)^r \\ &= {}^{12}C_r x^{24-2r} (-1)^r (y)^r (x)^r \\ &= {}^{12}C_r x^{24-r} y^r (-1)^r \end{aligned}$$

Question-19

Find the 4th term in the expansion of $(x - 2y)^{12}$.

Solution:

$$\begin{aligned} \text{4th term in } (x - 2y)^{12} &= T_4 = T_{3+1} \\ &= {}^{12}C_3 (x)^{12-3} (-2y)^3 \\ [T_{r+1} \text{ in } (a+b)^n &= {}^nC_r a^{n-r} b^r] \\ &= {}^{12}C_3 (x)^9 (-2)^3 (y)^3 \\ &= \frac{12 \times 11 \times 10}{3 \times 2} \times -8 \times x^9 y^3 \\ &= -1760 x^9 y^3 \end{aligned}$$

Question-20

Find the 13th term in the expansion of $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$

Solution:

$$\begin{aligned}13^{\text{th}} \text{ term in } \left(9x - \frac{1}{3\sqrt{x}}\right)^{18} &= T_{13} = T_{12+1} \\&= {}^{18}C_{12}(9x)^{18-12}\left(-\frac{1}{3\sqrt{x}}\right)^{12} \\&= {}^{18}C_{12}(9x)^6\left(-\frac{1}{3}\right)^{12}\left(-\frac{1}{\sqrt{x}}\right)^{12} \\&= {}^{18}C_{12}(9x)^6\left(-\frac{1}{3}\right)^{12}\left(x^{-\frac{1}{2}}\right)^{12} \\&= {}^{18}C_{12}(9x)^6\left(-\frac{1}{3}\right)^{12}(x)^{-6} \\&= {}^{18}C_{12}(3^2)^6(x)^6(-1)^{12}(3)^{-12}(x)^{-6} \\&= {}^{18}C_{12}(3^{12})(x)^6(3)^{-12}(x)^{-6} \\&= {}^{18}C_{12} = 18564\end{aligned}$$

Question-21

Find the 13th term in the expansion of $\left(3 - \frac{x^3}{6}\right)^7$

Solution:

The index of $\left(3 - \frac{x^3}{6}\right)^7$, is 7, which is an odd natural number.

So, Middle terms are $\frac{T_{7+1}}{2}$ and $\frac{T_{7+3}}{2}$

$$\begin{aligned}\frac{T_{7+1}}{2} &= T_4 = T_{3+1} = {}^7C_3(3)^{7-3}\left(-\frac{x^3}{6}\right)^3 \\&= {}^7C_3(3)^4(x^3)^3(-1)(6^{-1})^3 \\&= {}^7C_3(81)(x)^9(6)^{-3}(-1)^3 \\&= \frac{-35 \times 81}{216} \times 9 = \frac{-105}{8} \times 9 \\\frac{T_{7+3}}{2} &= T_5 = T_4 + 1 = {}^7C_4(3)^{7-4}\left(-\frac{x^3}{6}\right)^4 \\&= {}^7C_4(3)^3(-1)^4(x^3)(6)^{-4} \\&= {}^7C_4(27)(x)^{12}(6)^{-4} \\&= (35)(27)(6)^{-4}(x)^{12} \\&= \frac{35 \times 27}{1296} \times 12 = \frac{35}{48} \times 12\end{aligned}$$

Question-22

$$\left(\frac{x}{3} + 9y\right)^{10}$$

Solution:

The index $\left(\frac{x}{3} + 9y\right)^{10}$ is 10, which is an even natural number.

$$\begin{aligned}\text{Hence, Middle term} &= \frac{T_{10+2}}{2} = T_6 = T_{5+1} \\&= {}^{10}C_5 \left(\frac{x}{3}\right)^{10-5} (9y)^5 \\&= {}^{10}C_5 \left(\frac{x}{3}\right)^5 (9y)^5 \\&= {}^{10}C_5 (x)^5 \left(\frac{1}{3}\right)^5 (9)^5 (y)^5 \\&= {}^{10}C_5 (3)^{-5} (3^2)^5 (x)^5 (y)^5 \\&= {}^{10}C_5 (3)^{-5} (3)^{10} x^5 y^5 \\&= {}^{10}C_5 3^5 x^5 y^5 = (252)(243)x^5 y^5 \\&= 61236 x^5 y^5\end{aligned}$$

Question-23

In the expansions of $(1 + a)^{m+n}$, using Binomial Theorem, prove that coefficients of a^m and a^n are equal.

Solution:

We have,

$$(1 + a)^{m+n} = {}^{m+n}C_0 + {}^{m+n}C_1 a^1 + {}^{m+n}C_2 a^2 + \dots + {}^{m+n}C_r a^r + \dots + {}^{m+n}C_{m+n} a^{m+n}$$

$$\text{Coefficient of } a^m = {}^{m+n}C_m = \frac{(m+n)!}{m!n!}$$

Also the coefficient of a^n

$$= {}^{m+n}C_n = \frac{(m+n)!}{n!m!}$$

Clearly, ${}^{m+n}C_m = {}^{m+n}C_n$

Question-24

The coefficients of the $(r-1)^{\text{th}}$, r^{th} and $(r+1)^{\text{th}}$ terms in the expansion of $(x+1)^n$ are in the ratio 1:3:5. Find both n and r.

Solution:

Coefficient of $(r-1)^{\text{th}}$ term = $C(n, r-2)$

Coefficient of r^{th} term = $C(n, r-1)$

Coefficient of $(r+1)^{\text{th}}$ term = $C(n, r)$

Considering 1st and 2nd

$$\frac{C(n, r-2)}{C(n, r-1)} = \frac{n!}{(r-2)!(n-r+2)!} \times \frac{(r-1)!(n-r+1)!}{n!} = \frac{(r-1)!(n-r+1)!}{(r-2)!(n-r+2)(n-r+1)!} = \frac{(r-1)(r-2)!}{(r-2)!(n-r+2)} = \frac{(r-1)}{n-r+2} = \frac{1}{3}$$

$$3r - 3 = n - r + 2$$

$$n - 4r = -5 \text{ -----(1)}$$

Considering 2nd and 3rd

$$\frac{C(n, r-1)}{C(n, r)} = \frac{3}{5} = \frac{n!}{(r-1)!(n-r+1)!} \times \frac{r!(n-r)!}{n!} = \frac{(n-r)!r(r-1)!}{(n-r+1)(n-r)!(r-1)!} = \frac{r}{n-r+1} = \frac{3}{5}$$

$$5r = 3n - 3r + 3$$

$$3n - 8r = -3 \text{ -----(2)}$$

$$2(n - 4r = -5)$$

$$2n - 8r = -10 \text{ -----(3)}$$

Subtract (3) from (2)

$$n = 7$$

Substitute $n = 7$ in (2)

We get $r = 3$

$$n = 7, r = 3$$

Question-25

Prove that the coefficient of x^n in the expansion of $(1+x)^{2n}$ is twice the coefficient of x^n in the expansion $(1+x)^{2n-1}$.

Solution:

$$(1+x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1x^1 + {}^{2n}C_2x^2 + \dots + {}^{2n}C_nx^n$$

$$(1+x)^{2n-1} = {}^{2n-1}C_0 + {}^{2n-1}C_1x^1 + {}^{2n-1}C_2x^2 + \dots + {}^{2n-1}C_nx^n$$

Coefficient of x^n in $(1+x)^{2n-1}$ is $({}^{2n-1}C_n)$

$$\begin{aligned} {}^{2n}C_n &= \frac{(2n)!}{n!n!} \\ &= \frac{(2n)(2n-1)(2n-2)!}{n(n-1)!n!} \\ &= \frac{(2n)(2n-1)2(n-1)!}{n(n-1)!n!} \\ &= 2 \left[\frac{2(2n-1)}{n!} \right] \dots\dots(i) \end{aligned}$$

$$\text{Now } {}^{2n-1}C_n = \frac{(2n-1)!}{n!(2n-1-n)!}$$

$$\begin{aligned} &= \frac{(2n-1)!}{n!(n-1)!} \\ &= \frac{(2n-1)(2n-2)!}{n!(n-1)!} \\ &= \frac{(2n-1)2(n-1)!}{n!(n-1)!} \\ &= \frac{2(2n-1)}{n!} \dots\dots(ii) \end{aligned}$$

From (i) and (ii) we have

$${}^{2n}C_n = 2 \cdot {}^{2n-1}C_n$$